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LETTERS TO THE EDITOR

NULL SPACE SOLUTION OF JORDAN CHAINS FOR DEROGATORY EIGENPROBLEMS

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1. INTRODUCTION

Golub and Wilkinson declared in 1976 [1] that, from the standpoint of classical algebra, the algebraic eigenvalue problem, EVP, has been completely solved, yet, many existing algorithms cannot handle defective EVPs efficiently and they recommended two methods based on singular value decomposition for the complete solution of a defective derogatory system. The associated vectors obtained by the singular decomposition methods cannot produce the Jordan block structure. Golub would not now stand behind his 1976 statement but stressed the difficulty in the numerical solution of the derogatory eigenproblem [2]. In this letter a new and a much simpler nullspace method to solve the derogatory system completely is recommended. The associated vectors are required to be renormalized and to be rearranged in order to produce the final Jordan canonical form. A detailed numerical example is given.

The eigenvalue problem (EVP) associated with a damped dynamic system is quadratic: $(\lambda^2[\mathbf{M}] + \lambda[\mathbf{C}] + [\mathbf{K}])\{\mathbf{x}\} = \{\mathbf{0}\}$. The eigensolutions can be defective, i.e., there are more eigenvalues than eigenvectors. Symmetric polynomial EVPs as well as linear non-symmetric EVPs can be defective. There is no efficient algorithm to solve large scale defective EVPs.

The number of equilibrium solutions of a non-linear structure will change when the determinant of the tangential stiffness matrix equals zero. The directions of the new equilibrium solutions are determined by the eigenvectors associated with the tangential stiffness matrix which are called the null space. The null space can also be defective, i.e., there are more zero eigenvalues than the eigenvectors and associated (auxiliary, or generalized) vectors are required to span the null space.

The Jacobian matrix of non-linear aerodynamic flutter corresponding to the onset of a limit cycle constitutes a defective EVP. The exactly defective Jacobian is obtained by adjusting some parameters, such as speed and flap angle. Once the

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defective EVP is solved, the amplitude and stability of the limit cycle as functions of speed and flap angle can be studied by the central manifold and normal form theories [3]. When two or more limit cycles are coincident, a derogatory EVP results. In control theory, the observability condition requires the solution of a degenerate EVP. The complete solution of a derogatory EVP is also useful in the classification of degenerate systems in catastrophe and normal form theories.

A few common examples of defective engineering systems are given below.

(i) The symmetric complex EVP [4]:

$$[\mathbf{C}]\{\mathbf{x}\} = \lambda\{\mathbf{x}\} \quad \text{where} \quad [\mathbf{C}] = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

with $(a-b)^2 + 4c^2 = 0$ which has a defective eigensolution at $\lambda = (a+b)/2$.

(ii) The generalized EVP associated with positive definite damping [5]:

$$\{\ddot{\mathbf{x}}\} + \begin{bmatrix} 1 & \sqrt{15}/2 \\ \sqrt{15}/2 & 4 \end{bmatrix}\{\dot{\mathbf{x}}\} + \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}\{\mathbf{x}\} = \{\mathbf{0}\}$$

has a two-fold defective eigensolution at $\lambda = -2$.

(iii) The generalized EVP associated with negative definite damping:

$$\{\ddot{\mathbf{x}}\} - \frac{1}{5}\begin{bmatrix} 8 & 9 \\ 9 & 32 \end{bmatrix}\{\dot{\mathbf{x}}\} + \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}\{\mathbf{x}\} = \{\mathbf{0}\}$$

has a four-fold defective eigensolution at $\lambda = -2$.

The exactly derogatory EVP is dealt with in this note and another report will be devoted to the perturbed problems similar to reference [6].

After introducing the necessary terminology, the null space method of finding the grade 1 to 4 vectors is outlined. Grade 1 vectors are the eigenvectors and vectors of higher grades are associated vectors which are required to span the supplementary space of the eigenvectors. Since the determination of vectors of grades 1 to 3 differ slightly, they will be considered separately. Vectors of grades 4 and higher are then found routinely. The associated vectors are required to be renormalized and to be rearranged in order to produce the final Jordan canonical form. A detailed numerical example is given.

2. DEFINITIONS AND PROPERTIES

The mathematician Jordan [7] must have the full credit of the Jordan form structure. Sixty years ago, Frazer *et al*. [8] recognized that, by means of similarity transformation, a square matrix [**A**] can be reduced to the simplest canonical Jordan form [**J**]. The exact type of the canonical matrix is specified by means of its Segre characteristic. For example, the Segre characteristic [8] of

$$
[\mathbf{J}] = \begin{bmatrix}
\alpha & 1 & & & & & & & & \\
 & \alpha & & & & & & & & \\
 & & \alpha & & & & & & & \\
 & & & \alpha & 1 & & & & & \\
 & & & & \alpha & 1 & & & & \\
 & & & & & \alpha & & & & \\
 & & & & & & \beta & 1 & & \\
 & & & & & & & \beta & & \\
 & & & & & & & & \gamma & \\
 & & & & & & & & & \delta
\end{bmatrix}
$$

is [(213)211]. The three Jordan blocks of the same eigenvalue are bracketed together to form three blocks. The second block consists of one grade 1 vector only and the first and third blocks consist of grades 1 and 2 and grades 1, 2 and 3 vectors, respectively. Vectors of grade 1 $\{\mathbf{x}_1\}$ correspond to eigenvectors and vectors of other grades $\{\mathbf{x}_i\}$, $i \neq 1$, which are not eigenvectors, are obtained from the sequence: $[\mathbf{B}]\{\mathbf{x}_1\} = \{\mathbf{0}\}$, $[\mathbf{B}]\{\mathbf{x}_2\} = \{\mathbf{x}_1\}$, $[\mathbf{B}]\{\mathbf{x}_3\} = \{\mathbf{x}_2\}$, etc., where $[\mathbf{B}] = [\mathbf{A}] - \alpha[\mathbf{I}]$ is the shifted matrix. The vectors thus obtained are called the associated vectors of eigenvalue α. The associated vectors of block k whose Jordan block has dimension n_k are called the n_k vectors of chain k. A much fuller reference on canonical forms than the famous book by Frazer *et al.* is reference [9].

When all eigenvalues are distinct, one has a simple EVP, the Segre characteristic consists of n ones, where n is the order of the matrix and each chain consists of one vector which is the eigenvector. Let n_a eigenvalues be identically equal to α, or let the algebraic multiplicity of α be n_a and let n_g be the total number of eigenvectors associated with α or let n_g be the geometric multiplicity of α. Note the fundamental inequality $n_g \leqslant n_a$. There are two combinational properties of the generalized vectors.

Property (A). If $n_a = n_g$, one has a semi-simple EVP which has n_a-multiple eigenvalues and the same number of eigenvectors. The Segre characteristic consists of n_g ones within brackets. If $[\mathbf{X}]$ is the collection of n_g eigenvectors, so is $[\mathbf{X}][\mathbf{C}]$, where $[\mathbf{C}]$ is an arbitrary non-zero constant square matrix of order n_g, because if $[\mathbf{B}][\mathbf{X}] = [\mathbf{0}]$ is true so is $[\mathbf{B}][\mathbf{X}][\mathbf{C}] = [\mathbf{0}]$. That is, all linear combinations of eigenvectors are also eigenvectors.

Property (B). If there is only one Jordan block corresponding to α, or α is of block 1, it is necessary that $n_g = 1$ with only one eigenvector $\{\mathbf{x}_1\}$. If $[\mathbf{X}] = [\mathbf{x}_i\colon i = 1, \ldots, n_a]$ are the associated vectors, so are $[\mathbf{X}][\mathbf{\Delta}]$, where the lower triangular square matrix $[\mathbf{\Delta}]$ of order n_a has the form, for example $n_a = 3$,

$$
[\mathbf{\Delta}] = \begin{bmatrix} a & & \\ b & a & \\ c & b & a \end{bmatrix}, \tag{1}
$$

in which $a \neq 0$, b, c are arbitrary constants. It is shown below.

$$
[\mathbf{B}]\{a\,\mathbf{x}_1\} = [\mathbf{A} - \alpha\mathbf{I}\{a\,\mathbf{x}_1\} = \{\mathbf{0}\},
$$

$$[\mathbf{B}]\{\mathbf{x}_2 + p\mathbf{x}_1\} = \{\mathbf{x}_1\},$$

or

$$[\mathbf{B}]\{a\mathbf{x}_2 + b\mathbf{x}_1\} = \{a\mathbf{x}_1\},$$

where a, b, p are arbitrary numbers and $b = ap$. Equation (1) is the general form for three vectors.

When there are n_s Jordan blocks associated with α, the geometric multiplicity $n_g = n_s$. Let the Jordan chains be $[\mathbf{X}_k]$, $k = 1, \ldots, n_s$ and, as usual, the first vector of each chain is the eigenvector. Therefore, $[\mathbf{X}_k][\boldsymbol{\Delta}_k]$ is also the corresponding Jordan chain where $[\boldsymbol{\Delta}_k]$ is a lower triangular matrix of order n_k in the form (1). Since the linear combinations of the n_s eigenvectors are also eigenvectors, the linear combinations of the n_s chains $[\mathbf{X}_k][\boldsymbol{\Delta}_k]$ are also Jordan chains.

3. GRADE 1 VECTORS

With an accurate eigenvalue α of $[\mathbf{A}]$ which has been determined using either EISPACK [10] or the Lanczos method with inverse iteration improvement [11], its Segre characteristic and the associated Jordan chains will be found in the following sections.

The shifted matrix $[\mathbf{B}] = [\mathbf{A}] - \alpha[\mathbf{I}]$ of order n is rank deficient. The degree of deficiency n_x can be determined by the method of singular value decomposition (SVD) [12]. The number of zero (or very small) singular values is equal to the required n_x. The grade 1 vectors, eigenvectors, can be determined by the null space of $[\mathbf{B}]$, from the partitioned equation,

$$\begin{bmatrix} \mathbf{B}_{00} & \mathbf{B}_{01} \\ \mathbf{B}_{10} & \mathbf{B}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{I} \end{bmatrix} = [\mathbf{0}], \tag{2}$$

where $[\mathbf{I}]$ is an identity matrix of order n_x and $[\mathbf{X}^{\mathrm{T}}, \mathbf{I}]^{\mathrm{T}}$ are the required n_x eigenvectors, whose linear combinations are also eigenvectors. If the eigenvectors are $[\mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2^{\mathrm{T}}]^{\mathrm{T}}$, so are $[\mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2^{\mathrm{T}}]^{\mathrm{T}}[\mathbf{C}]$, where $[\mathbf{C}]$ is an arbitrary non-singular square matrix. Let $[\mathbf{C}] = [\mathbf{X}_2]^{\mathrm{T}}$, one has $[\mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2^{\mathrm{T}}]^{\mathrm{T}}][\mathbf{C}] = [(\mathbf{X}_1\mathbf{X}_2^{-1})^{\mathrm{T}}, \mathbf{I}] = [\mathbf{X}^{\mathrm{T}}, \mathbf{I}]^{\mathrm{T}}$, where $\mathbf{X} = \mathbf{X}_1\mathbf{X}_2^{-1}$. Therefore, the form $[\mathbf{X}^{\mathrm{T}}, \mathbf{I}]^{\mathrm{T}}$ is always possible. From equation (2), $[\mathbf{X}]$ can be easily solved from

$$[\mathbf{B}_{00}][\mathbf{X}] = -[\mathbf{B}_{01}] \tag{3}$$

because the matrix $[\mathbf{B}_{00}]$ is well-conditioned and has been decomposed in the checking of rank deficiency by SVD. The algorithm requires no knowledge of other eigensolutions; it is efficient.

4. GRADE 2 VECTORS

Because of the structure of $[\boldsymbol{\Delta}]$ in equation (1), one can always eliminate the last n_x row of the grade 2 vectors so that the grade 2 vectors are in the form

$[\mathbf{Y}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$, where $[\mathbf{0}]$ is a zero matrix of order $n_x \times n_y$, in which n_y is the number of the grade 2 vectors to be determined. Since the linear combinations of the eigenvectors are also eigenvectors, introduce the $n_x \times n_y$ combination matrix $[\mathbf{R}]$ which is to be determined from the following equation,

$$\begin{bmatrix} \mathbf{B}_{00} & \mathbf{B}_{01} \\ \mathbf{B}_{10} & \mathbf{B}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{Y} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} \\ \mathbf{I} \end{bmatrix} [\mathbf{R}], \tag{4}$$

or alternatively,

$$[\mathbf{B}_{10}\mathbf{B}_{00}^{-1}\mathbf{X} - \mathbf{I}][\mathbf{R}] = [\mathbf{0}] \tag{5}$$

after solving for $[\mathbf{Y}]$ in terms of $[\mathbf{X}]$ in the first equation of (4) and substituting into the second equation. That is, $[\mathbf{R}]$ is the null space of the square matrix equation (5) of order n_x. In practice, the matrix $[\mathbf{B}_{00}]$ is decomposed in the checking of rank deficiency by SVD for grade 1 vectors. Back substitution will give $[\mathbf{Y}_1] = [\mathbf{B}_{00}]^{-1}[\mathbf{X}]$ and the null space $[\mathbf{R}]$ together with its dimension n_y of $[\mathbf{B}_{10}\mathbf{Y}_1 - \mathbf{I}]$ are determined in exactly the same manner as in equation (2). If equation (5) has no null space, or the null space is empty, then the whole process of finding associated vectors is halted and one proceeds to section 7 for renormalization. Finally, $[\mathbf{Y}] = [\mathbf{Y}_1][\mathbf{R}]$ gives the n_y grade 2 vectors.

5. GRADE 3 VECTORS

Because of the structure of $[\boldsymbol{\Delta}]$ in equation (1), one can always eliminate the last n_x rows of the grade 3 vectors so that the grade 3 vectors are in the form $[\mathbf{Z}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$, where $[\mathbf{0}]$ is a zero matrix of order $n_x \times n_z$, in which n_z is the number of the grade 3 vectors to be determined. Since the linear combinations of the grade 1 and 2 vectors are also associated vectors, one can introduce the $(n_x + n_y) \times n_z$ combination matrix $[\mathbf{S}]$ which is to be determined from the following equation:

$$\begin{bmatrix} \mathbf{B}_{00} & \mathbf{B}_{01} \\ \mathbf{B}_{10} & \mathbf{B}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{Z} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} [\mathbf{S}], \tag{6}$$

or alternatively,

$$[\mathbf{B}_{10}\mathbf{B}_{00}^{-1}(\mathbf{X}, \mathbf{Y}) - (\mathbf{I}, \mathbf{0})][\mathbf{S}] = [\mathbf{0}]. \tag{7}$$

That is, $[\mathbf{S}]$ is the null space of the matrix equation (7). In practice, the matrix $[\mathbf{B}_{00}]$ is decomposed in the checking of rank deficiency by SVD for grade 1 vectors. Back substitution will give $[\mathbf{Z}_1] = [\mathbf{B}_{00}]^{-1}[\mathbf{X}, \mathbf{Y}]$ and the null space $[\mathbf{S}]$ together with its dimension n_z, of $[\mathbf{B}_{10}\mathbf{Z}_1 - (\mathbf{I}, \mathbf{0})]$ are determined in exactly the same manner as in equation (2). Some of the null vectors in the null space $[\mathbf{S}]$ do not contain any components of $[\mathbf{Y}]$ and these null vectors must be disregarded reducing $[\mathbf{S}]$ to $[\mathbf{S}_1]$, because these null vectors are simply other combinations of the columns of $[\mathbf{X}]$ which do not generate new associated vectors. The number of new associated vectors n_z must be reduced accordingly. If equation (7) has no null space containing components of $[\mathbf{Y}]$, then the whole process of finding associated vectors is halted and one proceeds to section 7 for renormalization. Finally, $[\mathbf{Z}] = [\mathbf{Z}_1][\mathbf{S}_1]$ gives the n_z grade 3 vectors.

6. GRADE 4 VECTORS AND VECTORS OF HIGHER GRADES

Because of the structure of $[\boldsymbol{\Delta}]$ in equation (1), one can always eliminate the last n_x row of the grade 4 vectors so that the grade 4 vectors are in the form $[\mathbf{W}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$, where $[\mathbf{0}]$ is a null matrix of order $n_x \times n_w$, in which n_w is the number of the grade 4 vectors to be determined. Since the linear combinations of the grades 1, 2 and 3 vectors are also associated vectors, one can introduce the $(n_x + n_y + n_z) \times n_w$ combination matrix $[\mathbf{T}]$ which is to be determined from the following equation,

$$\begin{bmatrix} \mathbf{B}_{00} & \mathbf{B}_{01} \\ \mathbf{B}_{10} & \mathbf{B}_{11} \end{bmatrix} \begin{bmatrix} \mathbf{W} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} & \mathbf{Z} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} [\mathbf{T}], \tag{8}$$

or alternatively,

$$[\mathbf{B}_{10}\mathbf{B}_{00}^{-1}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) - (\mathbf{I}, \mathbf{0}, \mathbf{0})][\mathbf{T}] = [\mathbf{0}]. \tag{9}$$

That is, $[\mathbf{T}]$ is the null space of the matrix equation (9). In practice, the matrix $[\mathbf{B}_{00}]$ is decomposed in the checking of rank deficiency by SVD. Back substitution will give $[\mathbf{W}_1] = [\mathbf{B}_{00}]^{-1}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}]$ and the null space $[\mathbf{T}]$ together with its dimension n_w of $[\mathbf{B}_{10}\mathbf{W}_1 - (\mathbf{I}, \mathbf{0}, \mathbf{0})]$ are determined in exactly the same manner as in equation (2). Some of the null vectors in the null space $[\mathbf{T}]$ do not contain any components of $[\mathbf{Z}]$ which must be disregarded, thus reducing $[\mathbf{T}]$ to $[\mathbf{T}_1]$, because these null vectors are alternative combinations of $[\mathbf{X}, \mathbf{Y}]$ which do not generate new associated vectors. The number of new associated vectors n_w must be reduced accordingly. If equation (9) has no null space containing components of $[\mathbf{Z}]$, then the whole process of finding associated vectors is halted and one can proceed to section 7 for renormalization. Finally, $[\mathbf{W}] = [\mathbf{W}_1][\mathbf{T}_1]$ gives the n_w grade 4 vectors.

The process can be augmented to associated vectors of higher grades.

7. RENORMALIZATION OF ASSOCIATED VECTORS

Let

$$[\mathbf{V}] = \begin{bmatrix} \mathbf{X} & \mathbf{Y} & \mathbf{Z} & \mathbf{W} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

be the collection of all the computed associated vectors. Instead of the Jordan canonical form for the zero eigenvalue of $[\mathbf{B}]$,

$$[\mathbf{B}][\mathbf{U}] = [\mathbf{U}][\mathbf{J}] \tag{10}$$

one may just have (since the properties (A) and (B) of section 2 are used separately one at a time, but not the combination)

$$[\mathbf{B}][\mathbf{V}] = [\mathbf{V}][\mathbf{K}], \tag{11}$$

where the matrix $[\mathbf{K}]$ can be determined from a generalized inverse, in the absence of the right associated vectors, e.g.,

$$[\mathbf{K}] = [\mathbf{V}^{\mathrm{T}}\mathbf{V}]^{-1}[\mathbf{V}]^{\mathrm{T}}[\mathbf{B}][\mathbf{V}]. \tag{12}$$

If there is a transformation matrix $[\mathbf{P}]$, such that $[\mathbf{U}] = [\mathbf{V}][\mathbf{P}]$ will transform equation (11) to (10), then, matrix $[\mathbf{P}]$ must satisfy

$$[\mathbf{K}][\mathbf{P}] = [\mathbf{P}][\mathbf{J}] \tag{13}$$

or, $[\mathbf{P}]$ is the associated matrix of the graded vectors. Since the order of $[\mathbf{K}]$ is rather small, one apparent solution is to extract $[\mathbf{P}]$ from the null space of $[\mathbf{K}]^p$. Unfortunately, the graded vectors obtained are all unit vectors and the transformation to Jordan canonical form is not possible.

$[\mathbf{P}]$ is obtained by the Gauss elimination method and similarity transformation in the following manner. Both matrices $[\mathbf{K}]$ and $[\mathbf{J}]$ are strictly upper triangular having zero diagonal terms associated with the shifted matrix $[\mathbf{B}]$. The upper sub-diagonal of matrix $[\mathbf{J}]$ has entries one or zero. The strict upper triangle of $[\mathbf{K}]$ contains the same number of ones in the appropriate positions and contains some additional non-zero numbers which are to be eliminated. Elementary row operations $[\mathbf{Q}_1]$ are performed on the augmented matrix $[\mathbf{K}, \mathbf{I}]$ to eliminate the unwanted entries of $[\mathbf{K}]$, that is,

$$[\mathbf{Q}_1][\mathbf{K}, \mathbf{I}] = [\mathbf{Q}_1\mathbf{K}, \mathbf{Q}_1] = [\mathbf{L}, \mathbf{Q}_1], \tag{14}$$

where $[\mathbf{L}] = [\mathbf{Q}_1\mathbf{K}]$ contains no unwanted entries and has the same number as $[\mathbf{J}]$ but in different positions. Then perform the similarity transformation,

$$[\mathbf{K}_1] = [\mathbf{Q}_1\mathbf{K}\mathbf{Q}_1^{-1}]. \tag{15}$$

Since $\mathbf{Q}_1$ contains only a small number of elementary row operations, the inversion involves the same number of elementary operations. The process is equivalent to a QR transformation. All the unwanted entries generated by grade 1 vectors are eliminated in $[\mathbf{K}_1]$. One can perform the transformation m times, where $m+1$ is the maximum number of grades, then $[\mathbf{K}_m] = [\mathbf{L}]$ which has the same structure as $[\mathbf{J}]$.

The matrix $[\mathbf{L}]$ has the same structure as $[\mathbf{J}]$ except that the off-diagonal entries are in different positions. The transformation from $[\mathbf{L}]$ to $[\mathbf{J}]$ can be achieved by the permutation matrix $[\mathbf{N}]$, so that $[\mathbf{N}][\mathbf{L}][\mathbf{N}]^{-1} = [\mathbf{J}]$. If one wishes to re-order the diagonal entries of $[\mathbf{L}]$ in the sequence $\{s_1, s_2, \ldots, s_m\}$, where m is the algebraic multiplicity of α, then, the ith column of the permutation matrix $[\mathbf{N}]$ is given by $\{\mathbf{e}_{si}\}$ which is the s_ith unit vector. Therefore,

$$[\mathbf{P}] = [\mathbf{Q}_m][\mathbf{Q}_{m-1}]\ldots[\mathbf{Q}_2][\mathbf{Q}_1][\mathbf{N}]^{-1}. \tag{16}$$

Finally the required associated vectors are given by

$$[\mathbf{U}] = [\mathbf{V}][\mathbf{P}]. \tag{17}$$

8. PERTURBATION AND ERROR ANALYSIS

If an EVP is a perturbation of a derogatory EVP, one of the solution methods is to solve the derogatory EVP completely first and find the perturbed solutions according to reference [13]. Unfortunately, it is not always possible to find the derogatory EVP itself. In such a case, one can try to solve the EVP using EISPACK and check the dependency of the computed eigenvectors associated with an approximately multiple eigenvalue by SVD similar to the rank determination to guess the Jordan structure.

The numerical example in the next section is based on exact arithmetics so that the rank determination by SVD for each group of graded vectors is exact. If floating point arithmetics with finite precision is used, the ratios of the singular values must be checked before replacing the smallest singular values with zeros for rank determination, according to chapters 2 and 12 of reference [2].

The error analysis is a subject of its own. A more mathematical paper will be prepared for the full error analysis of the present method based on an extended matrix formulation [14].

9. NUMERICAL EXAMPLE

Construct [**A**] from the following matrices,

$$[\mathbf{J}] = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$[\mathbf{U}] = \begin{bmatrix} 3432 & 1716 & 792 & 330 & 120 & 36 & 8 & 1 \\ 1716 & 924 & 462 & 210 & 84 & 28 & 7 & 1 \\ 792 & 462 & 252 & 126 & 56 & 21 & 6 & 1 \\ 330 & 210 & 126 & 70 & 35 & 15 & 5 & 1 \\ 120 & 84 & 56 & 35 & 20 & 10 & 4 & 1 \\ 36 & 28 & 21 & 15 & 10 & 6 & 3 & 1 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

to obtain $[\mathbf{A}] = [\mathbf{U}][\mathbf{J}][\mathbf{U}]^{-1} =$

$$\begin{bmatrix} -27\,673 & 195\,490 & -588\,314 & 976\,354 & -963\,056 & 562\,924 & -179\,720 & 24\,004 \\ -14\,505 & 101\,795 & -303\,926 & 499\,450 & -486\,458 & 279\,580 & -87\,164 & 11\,236 \\ -6867 & 47\,739 & -140\,867 & 228\,018 & -217\,644 & 121\,590 & -36\,330 & 4368 \\ -2820 & 19\,305 & -55\,840 & 87\,991 & -80\,850 & 42\,630 & -11\,550 & 1140 \\ -929 & 6167 & -17\,097 & 25\,319 & -21\,069 & 9254 & -1582 & -58 \\ -200 & 1207 & -2865 & 3139 & -1016 & -1049 & 1078 & -290 \\ -3 & -60 & 486 & -1455 & 2277 & -2002 & 946 & -186 \\ 12 & -104 & 386 & -797 & 991 & -744 & 313 & -55 \end{bmatrix}.$$

It is interesting to note that [**U**] is an inverted Pascal triangle whose inverse is also an integer matrix.

The eigenvalues of [**A**] are computed using EISPACK and it is found that there is a six-fold multiple eigenvalue $\alpha = 1$ and the other two eigenvalues are 6 and 7. EISPACK is not able to compute the corresponding associated vectors. Therefore, section 3 is followed to obtain an unknown number of grade 1 vectors. Form $[\mathbf{B}] = [\mathbf{A}] - \alpha[\mathbf{I}]$ and find, from SVD of [**B**] that [**B**] has rank deficiency $n_x = 3$. The three grade 1 vectors $[\mathbf{X}^{\mathrm{T}}, \mathbf{I}]^{\mathrm{T}}$ are obtained from equation (3), $[\mathbf{B}_{00}][\mathbf{X}] = -[\mathbf{B}_{01}]$, as

$$[\mathbf{X}] = \begin{bmatrix} 210 & -798 & 1170 \\ 322/3 & -392 & 560 \\ 49 & -168 & 231 \\ 19 & -58 & 75 \\ 17/3 & -13 & 15 \end{bmatrix}.$$

To obtain the unknown number n_y of grade 2 vectors, one gets from equation (5)

$$[\mathbf{B}_{10}\mathbf{B}_{00}^{-1}\mathbf{X} - \mathbf{I}] = \begin{bmatrix} 2/75 & -3/25 & 1/5 \\ 64/675 & -32/75 & 32/45 \\ 98/675 & -49/75 & 49/45 \end{bmatrix},$$

whose null space of order $n_y = 2$ is

$$[\mathbf{R}] = \begin{bmatrix} -15/2 & 9/2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$[\mathbf{Y}] = \begin{bmatrix} -553 & 119 \\ -511/2 & 329/6 \\ -201/2 & 43/2 \\ -61/2 & 13/2 \\ -11/2 & 7/6 \end{bmatrix},$$

where the grade 2 associated vectors are $[\mathbf{Y}^{\mathrm{T}}, \mathbf{0}^{\mathrm{T}}]^{\mathrm{T}}$.

To obtain the grade 3 vectors, from equation (7), evaluate

$$[\mathbf{B}_{10}\mathbf{B}_{00}^{-1}(\mathbf{X},\mathbf{Y})-(\mathbf{I},\mathbf{0})]=\begin{bmatrix}2/75 & -3/25 & 1/5 & -16/175 & 3/175\\ 64/675 & -32/75 & 32/45 & -111/350 & 551/9450\\ 98/675 & -49/75 & 49/45 & -47/100 & 227/2700\end{bmatrix},$$

with null space

$$[\mathbf{S}]=\begin{bmatrix}-15/2 & 1/2 & 9/2\\ 0 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1/3 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

Since columns 1 and 3 do not have contributions from [**Y**], they are discarded to obtain $n_z = 1$ and

$$[\mathbf{Z}]=\begin{bmatrix}-266/3\\ -257/6\\ -107/6\\ -35/6\\ -7/6\end{bmatrix}.$$

Since EISPACK indicates that there is a six-fold multiple eigenvalue $\alpha = 1$, and six associated vectors have been found already, one stops. If the multiplicity is not known beforehand, then one would proceed and stop when $n_w = 0$.

Using

$$[\mathbf{V}]=\begin{bmatrix}\mathbf{X} & \mathbf{Y} & \mathbf{Z}\\ \mathbf{I} & \mathbf{0} & \mathbf{0}\end{bmatrix}$$

as the associated vectors, it is found from equation (11) that

$$[\mathbf{K}]=\begin{bmatrix}0 & 0 & 0 & -15/2 & 9/2 & 1/2\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1/3\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

which has four non-zero entries above the ones and has to be eliminated by renormalization. After elementary row operations according to equation (14) on the augmented matrix

$$[\mathbf{K}, \mathbf{I}] = \begin{bmatrix} -\frac{15}{2} & \frac{9}{2} & \frac{1}{2} & & & & 1 & & & & & \\ & 1 & & & & & & 1 & & & & \\ 1 & & & & & & & & 1 & & & \\ & & \frac{1}{3} & & & & & & & 1 & & \\ & & 1 & & & & & & & & 1 & \\ & & & & & & & & & & & 1 \end{bmatrix},$$

which produces

$$\begin{bmatrix} & & & & & & 1 & -\frac{9}{2} & \frac{15}{2} & & -\frac{1}{2} & \\ & 1 & & & & & & 1 & & & & \\ 1 & & & & & & & & 1 & & & \\ & & & & & & & & & 1 & -\frac{1}{3} & \\ & & 1 & & & & & & & & 1 & \\ & & & & & & & & & & & 1 \end{bmatrix} \equiv [\mathbf{L}, \mathbf{Q}_1],$$

where the non-zero entries only are shown, one obtains, from equation (15),

$$[\mathbf{K}_1] = [\mathbf{Q}_1 \mathbf{K} \mathbf{Q}_1^{-1}] = \begin{bmatrix} 0 & & & & & \\ & 0 & & & 1 & \\ & & 0 & 1 & \frac{1}{3} & \\ & & & 0 & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix},$$

and one now has only one non-zero entry to process.

Again, from equation (14) on the augmented matrix $[\mathbf{K}_1, \mathbf{I}]$, it is found that

$$[\mathbf{Q}_2] = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & -\frac{1}{3} & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{K}_2] = [\mathbf{L}] = \begin{bmatrix} 0 & & & & & \\ & 0 & & & 1 & \\ & & 0 & 1 & & \\ & & & 0 & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}.$$

If the diagonal sequence of $[\mathbf{L}]$ is reordered to $\{1, 3, 4, 2, 5, 6\}$ using the permutation matrix $[\mathbf{N}] = [\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_6]$ where $\{\mathbf{e}_i\}$ is the ith unit vector, then one has the canonical Jordan form

$$[\mathbf{J}] = [\mathbf{N}][\mathbf{K}_2][\mathbf{N}]^{-1} = \begin{bmatrix} 0 & & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & 0 & & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}.$$

Finally the Segre characteristic is [(132)11] for the eigenvalues 1, 6 and 7 respectively.

10. CONCLUSION

A method has been presented to find the associated vectors of a derogatory EVP of any Jordan canonical when the eigenvalue α is given accurately by using the null space concept. The associated vectors are required to be renormalized and rearranged in order to produce the final Jordan canonical form. The matrix $[\mathbf{B}] = [\mathbf{A}] - \alpha[\mathbf{I}]$ is required to be decomposed only once by SVD.

REFERENCES

1. G. H. Golub and J. H. Wilkinson 1976 *SIAM Review* **18**, 578–619. Ill-conditioned eigensystems and the computation of the Jordan canonical form.
2. G. H. Golub and C. F. Van Loan 1996 *Matrix Computations*. Baltimore and London: Johns Hopkins; third edition.
3. Chen Yushu and A. Y. T. Leung 1998 *Bifurcation and Chaos in Engineering*. London: Springer.
4. A. Y. T. Leung 1995 *Journal of Sound and Vibration* **184**, 627–637. Subspace iteration for complex symmetric eigenvalue problems.
5. A. Y. T. Leung 1997 *Keynote paper, ASME Asian'97 Conference, Singapore*. Algorithms for large eigenvalue problems in vibration and buckling analyses.
6. A. Y. T. Leung 1990 *Communications in Applied Numerical Methods* **6**, 401–409. Perturbed general eigensolutions.
7. C. Jordan 1870 *Traite' des Substitutions*. Paris. See p. 114.
8. R. A. Frazer, W. J. Duncan and A. R. Collar 1947 *Elementary Matrices*. Cambridge: Cambridge University Press.
9. H. W. Tuenbull and A. C. Aitken 1932 *The Theory Canonical Matrices*. Blackie.
10. B. T. Smith, J. M. Boyle, J. J., Dongarra, B. S. Garbow, Y. Ikebe, V. C. Klema and C. B. Moler 1976 *Matrix Eigensystem Routines—EISPACK Guide*. Berlin: Springer.
11. A. Y. T. Leung, J. K. W. Chan and C. W. Chan 1998 *Journal of Vibration and Control* **11**, 351–355. Inverse iteration for large defective eigensystems.
12. W. H. Press, B. P. Flannery, S. A. Teukolsky and W. T. Vettering 1986 *Numerical Recipes*. Cambridge: Cambridge University Press.
13. J. Moro, J. V. Burke and M. L. Overton, 1997, *SIAM Journal of Matrix Analysis and Applications* **18**, 793–817. On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure.
14. S. C. Wong and A. Y. T. Leung 1998 *Communications in Numerical Methods in Engineering*, Determination of Jordan chains by extended matrices **14**, 879–893.